Stellarator Equilibrium Construction based on a Poincaré Sections Approach

Mechanical & Aerospace Engineering

Daniel Dudt, Egemen Kolemen (ddudt@pppl.gov)

Department of Mechanical & Aerospace Engineering, Princeton University

PPPL

Motivation

Conventional equilibria codes for non-axisymmetric geometries are computationally expensive and can return inaccurate solutions. As modern stellarator experiments continue to show promising results, it is becoming increasingly important to develop advanced equilibria codes for these devices. A new code is presented that directly solves the force balance equations using root-finding algorithms. This formulation also simplifies the search for neighboring equilibria solutions and may naturally provide stability analysis with minimal additional cost.

Ideal MHD Equilibrium

Under the ideal MHD model of plasmas, the equilibrium condition reduces to a momentum balance, Ampere's Law, and the absence of magnetic monopoles [1]:

$$\boldsymbol{J} \times \boldsymbol{B} = \nabla p \qquad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \qquad \nabla \cdot \boldsymbol{B} = 0$$

The existence of flux surfaces and the rotational transform profile can provide two additional constraints on the problem:

$$\boldsymbol{B} \cdot \nabla \rho = 0 \qquad \iota = \frac{B^{\vartheta}}{B^{\zeta}}$$

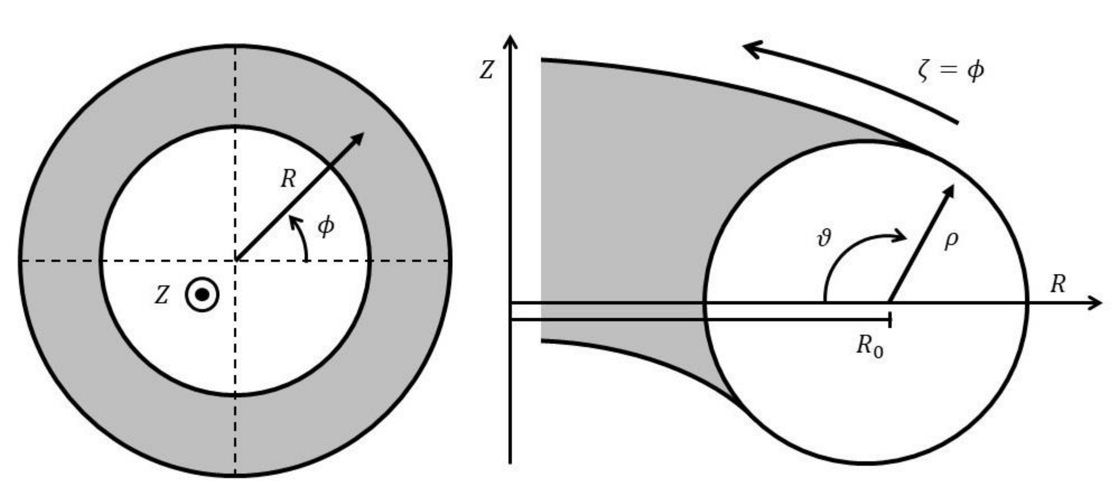
An equilibrium is typically constructed by prescribing profiles for the pressure $p(\rho)$ and rotational transform $\iota(\rho)$, then numerically solving for the magnetic field $\boldsymbol{B}$ that satisfies the equilibrium conditions. The most common approach used by conventional equilibria codes, including VMEC, is to solve for the field that minimizes the total energy of the plasma [2]:

$$W = \int \left(\frac{|B|^2}{2\mu_0} + p \right) dV$$

This method can be effective, but it does not solve the equilibrium force balance directly and is susceptible to numerical issues.

Straight Field Line Coordinates

The following diagram shows the toroidal (R, ϕ, Z) and straight field line (ρ, ϑ, ζ) coordinate systems [3]:



$$\sqrt{g} = \boldsymbol{e}_\rho \cdot \boldsymbol{e}_\vartheta \times \boldsymbol{e}_\zeta = R\left(R_\rho Z_\vartheta - R_\vartheta Z_\rho\right) \qquad \boldsymbol{e}_\alpha = \begin{bmatrix} R_\alpha \\ \phi_\alpha \\ Z_\alpha \end{bmatrix}$$

$$\rho = \sqrt{\Psi_N} \qquad \vartheta = \theta + \lambda$$

Poincaré Sections Approach

Instead of minimizing energy, the equilibrium force balance is used to derive a boundary value problem (BVP) that describes the "flow" of flux surfaces around the torus. Assuming nested flux surfaces and satisfying $\nabla \cdot \boldsymbol{B} = 0$, the magnetic field can be written with straight field line coordinates in contravariant form

$$\boldsymbol{B} = B^{\vartheta} \boldsymbol{e}_\vartheta + B^{\zeta} \boldsymbol{e}_\zeta = B^{\zeta}\left(\iota \boldsymbol{e}_\vartheta + \boldsymbol{e}_\zeta\right) \qquad B^{\zeta} = \frac{\Psi'}{2\pi\sqrt{g}}$$

Substituting this form of the magnetic field into the equilibrium equations, the force balance can be written as [2]

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \mu_0 \nabla p = \boldsymbol{0}$$

$$\boldsymbol{F} \equiv F_\rho \nabla p + F_\beta \boldsymbol{\beta} = \boldsymbol{0}$$

$$F_\rho = \frac{1}{\mu_0}\left(B^{\vartheta}\left(\partial_\rho B_\vartheta - \partial_\vartheta B_\rho\right) + B^{\zeta}\left(\partial_\rho B_\zeta - \partial_\zeta B_\rho\right)\right) + p'$$

$$F_\beta = \frac{1}{\mu_0 \sqrt{g}}\left(\partial_\vartheta B_\zeta - \partial_\zeta B_\vartheta\right) \qquad \boldsymbol{\beta} = \sqrt{g}\left(B^{\zeta} \nabla \vartheta - B^{\vartheta} \nabla \zeta\right)$$

Since ∇p and $\boldsymbol{\beta}$ are independent directions, $F_\rho = F_\beta = 0$ must hold in equilibrium. This yields the two constraint equations:

$$F_\rho = 0 \Rightarrow \partial_\zeta B_\rho = \iota\left(\partial_\rho B_\vartheta - \partial_\vartheta B_\rho\right) + \partial_\rho B_\zeta + \mu_0 \frac{p'}{B^{\zeta}}$$

$$F_\beta = 0 \Rightarrow \partial_\zeta B_\vartheta = \partial_\vartheta B_\zeta$$

The basis vectors are functions of R, Z and their partial derivatives, so these two equilibrium conditions can be rewritten into equations for $\ddot{R}(\rho, \vartheta, \zeta)$ and $\ddot{Z}(\rho, \vartheta, \zeta)$. By discretizing ρ and ϑ in the Poincaré sections, these become a system of ordinary differential equations (ODEs) with ζ as the canonical "time":

$$\boldsymbol{x}(\zeta) = \begin{bmatrix} R \\ Z \\ \lambda \\ \dot{R} \\ \dot{Z} \\ \dot{\lambda} \end{bmatrix} \quad \dot{\boldsymbol{x}} = \begin{bmatrix} \dot{R} \\ \dot{Z} \\ \dot{\lambda} \\ \frac{2\pi R}{\Psi'}\left(k_\rho Z_\vartheta - k_\beta Z_\rho\right) \\ -\frac{2\pi R}{\Psi'}\left(k_\rho R_\vartheta - k_\beta R_\rho\right) \\ \dfrac{\boldsymbol{e}_\vartheta \cdot \left(\boldsymbol{e}_{\theta\theta}\dot{\theta}^2 + 2\,\boldsymbol{e}_{\theta\phi}\dot{\theta} + \boldsymbol{e}_{\phi\phi}\right) - \frac{k_\beta}{B^{\zeta}}}{\boldsymbol{e}_\vartheta \cdot \boldsymbol{e}_\theta} \end{bmatrix} = f(\zeta, \boldsymbol{x})$$

$$k_\rho = \iota\left(\partial_\rho B_\vartheta - \partial_\vartheta B_\rho\right) + \partial_\rho B_\zeta + \mu_0 \frac{p'}{B^{\zeta}} - \partial_\zeta B^{\zeta}\left(\iota \boldsymbol{e}_\rho \cdot \boldsymbol{e}_\vartheta + \boldsymbol{e}_\rho \cdot \boldsymbol{e}_\zeta\right) - B^{\zeta}\left(\boldsymbol{e}_{\rho\zeta} \cdot \left(\iota \boldsymbol{e}_\vartheta + \boldsymbol{e}_\zeta\right) + \iota \boldsymbol{e}_\rho \cdot \boldsymbol{e}_{\vartheta\zeta}\right)$$

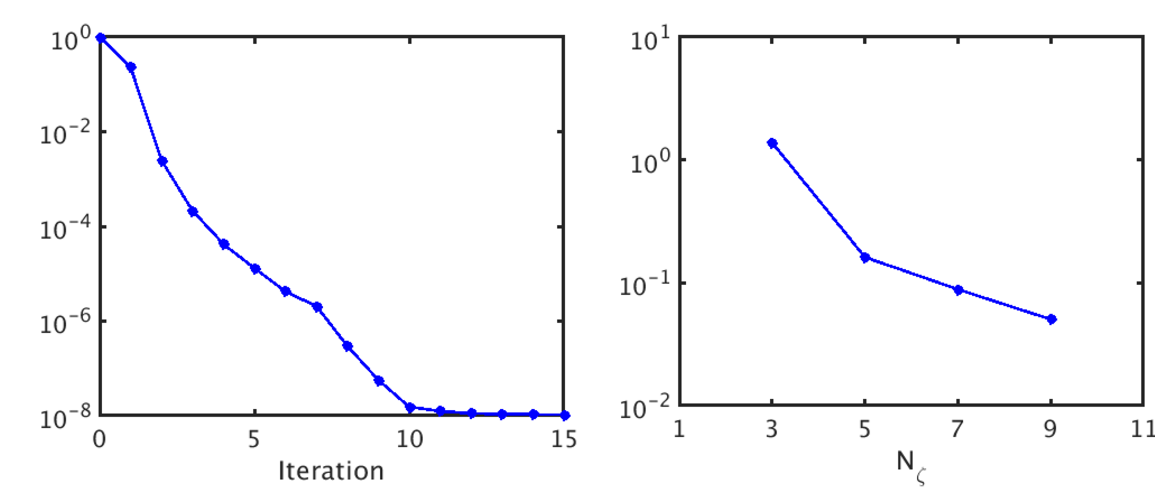
$$k_\beta = \partial_\vartheta B_\zeta - \partial_\zeta B^{\zeta}\left(\iota |\boldsymbol{e}_\vartheta|^2 + \left(\boldsymbol{e}_\vartheta \cdot \boldsymbol{e}_\zeta\right)\right) - B^{\zeta} \boldsymbol{e}_{\vartheta\zeta} \cdot \left(2\iota \boldsymbol{e}_\vartheta + \boldsymbol{e}_\zeta\right)$$

The variable λ is used instead of R and Z at the last closed flux surface where the fixed-boundary $R(\theta, \phi)$ and $Z(\theta, \phi)$ is given as a constraint. A periodic boundary condition arises since the equilibrium solution must represent the same flux surfaces after integrating over a full field period. Thus the equilibrium problem has been cast into the equivalent boundary-value problem:

$$\dot{\boldsymbol{x}}(\zeta) = f(\zeta, \boldsymbol{x}) \qquad \boldsymbol{x}(0) = \boldsymbol{x}(2\pi)$$

Comparison to VMEC

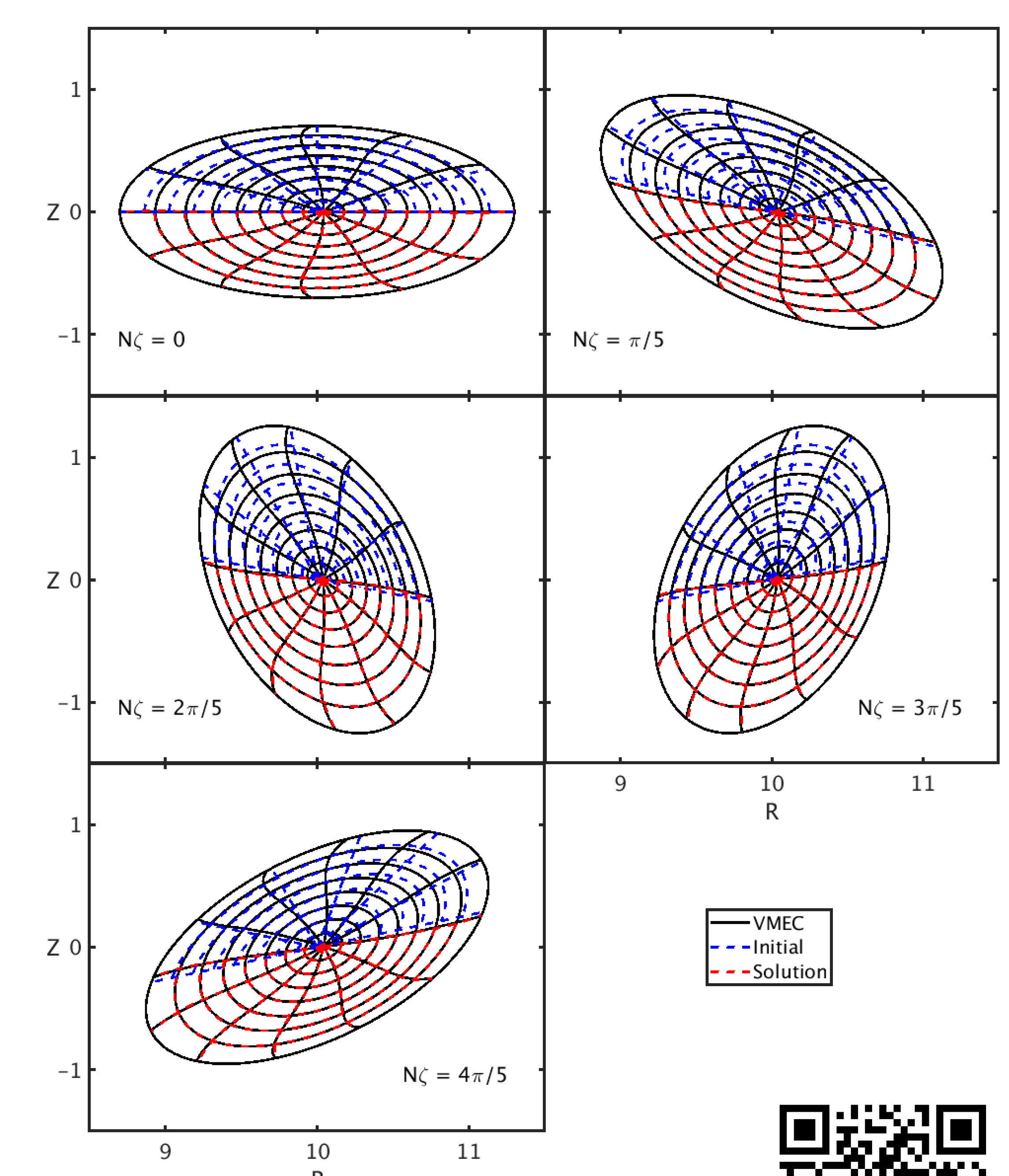
The computational domain is discretized using pseudo-spectral methods. For each Poincaré section, Fourier series in ϑ and Chebyshev polynomials in ρ were used in this case [4]. The BVP is solved using a spectral collocation approach with a Fourier series interpolant in ζ.



Left: Normalized sum of squares of the residuals at each iteration of the Levenberg–Marquardt algorithm used to solve the BVP.

Right: $|\boldsymbol{x}(0) - \boldsymbol{x}^{VMEC}(0)|$ as a function of the number of Fourier modes used in the toroidal expansion (number of Poincaré sections).

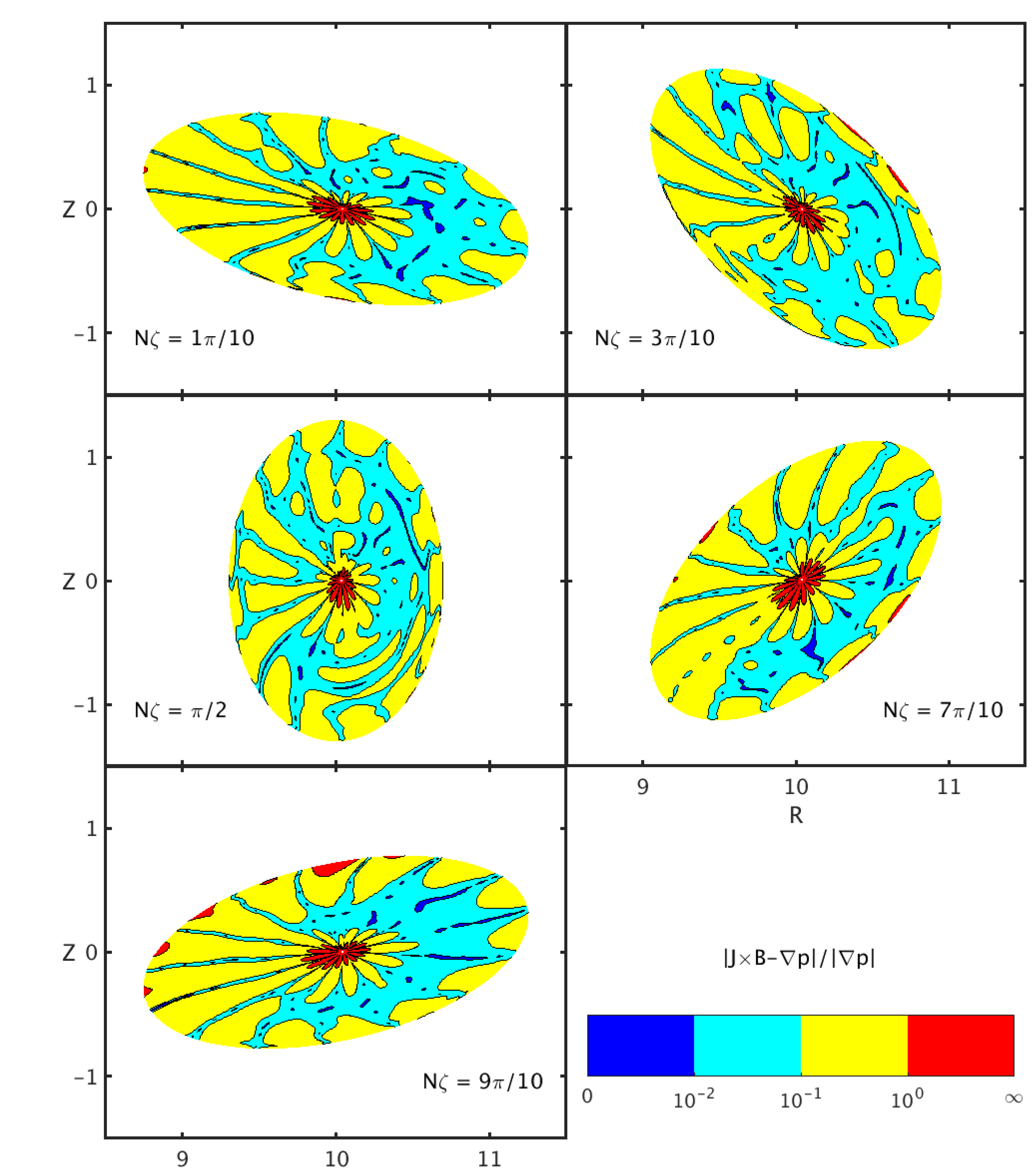
Below: Comparison of flux surfaces between the BVP solution and the VMEC equilibrium. The initial guess for the solution was obtained by scaling the boundary surface proportional to ρ.



Right: control.princeton.edu/presentations/

Error Quantification

Below: Equilibrium force balance error $|\boldsymbol{F}|/|\nabla p|$ for the BVP solution. The system of ODEs $\dot{\boldsymbol{x}} = f(\zeta, \boldsymbol{x})$ is equivalent to the equilibrium equations, so the approach directly minimizes the error at the collocation points. In other words, this pseudo-spectral method solves the equilibrium force balance at discrete points in space.



All the results presented are for a heliotron with the following boundary and profile inputs:

$$R = 10 - \cos\theta - 0.3\cos(\theta - N\phi) \qquad Z = \sin\theta - 0.3\sin(\theta - N\phi)$$

$$p(\rho) = p_0\left(1 - \rho^2\right) \qquad \iota(\rho) = 1.618$$

with the number of field periods $N = 19$ and p_0 corresponds to $\beta = 1.1\%$. A basis of 8 Fourier modes and 8 Chebyshev polynomials were used.

Future Work

- Refine the numerical methods to improve convergence and robustness to arbitrary inputs
- Investigate the existence of neighboring solutions through perturbation analysis
- Develop tools to assess the stability of equilibria
- Extend the formulation to include free-boundary equilibria
- Relax the constraint on nested flux surfaces to allow for magnetic islands and chaotic regions

[1] Freidberg, J. (2014) [2] Hirshman, S. P., & Whitson, J. C., (1983) [3] D'haeseleer, W. D., Hitchon, W. G., Callen, J. D., & Shohet, J. L. (1991) [4] Boyd, J. (2000)