Stellarator Equilibrium Construction based on a Poincaré Sections Approach

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Motivation
Conventional equilibria codes for non-axisymmetric geometries are computationally expensive and can return inaccurate solutions. As modern stellarator experiments continue to show promising results, it is becoming increasingly important to develop advanced equilibria codes for these devices. A new code is presented that directly solves the force balance equations using root-finding algorithms. This formulation also simplifies the search for neighboring equilibria solutions and may naturally provide stability analysis with minimal additional cost.

Ideal MHD Equilibrium
Under the ideal MHD model of plasmas, the equilibrium condition reduces to a momentum balance, Ampere’s Law, and the absence of magnetic monopoles [1]:

\[ J \times B = \nabla \times B = \mu_0 J \quad \nabla \cdot B = 0 \]

The existence of flux surfaces and the rotational transform profile can provide two additional constraints on the problem:

\[ B \cdot \nabla \rho = 0 \quad \mathbf{i} = \frac{\mathbf{B}^2}{\nabla B} \]

An equilibrium is typically constructed by prescribing profiles for the pressure \( p(r) \) and rotational transform \( \psi(r) \), then numerically solving for the magnetic field \( B \) that satisfies the equilibrium conditions. The most common approach used by conventional equilibria codes, including VMEC, is to solve for the field that minimizes the total energy of the plasma [2]:

\[ W = \int \left( \frac{\mathbf{B}^2}{2 \mu_0} + \rho \right) dV \]

This method can be effective, but it does not solve the equilibrium force balance directly and is susceptible to numerical issues.

Straight Field Line Coordinates
The following diagram shows the toroidal \((R, \phi, Z)\) and straight field line \((\rho, \theta, \lambda)\) coordinate systems [3]:

\[ B = B_{\rho} \mathbf{e}_\rho + B_{\lambda} \mathbf{e}_\lambda = B(\rho, \lambda, Z) \]

\[ \rho = \sqrt{\rho \mathbf{n} / \phi \mathbf{n} \mathbf{l}} \quad \theta = \theta + \lambda \]

\[ B = B(\rho, \lambda, Z) \]

Poincaré Sections Approach
Instead of minimizing energy, the equilibrium force balance is used to derive a boundary value problem (BVP) that describes the “flow” of flux surfaces around the torus. Assuming nested flux surfaces and satisfying \( \nabla \cdot B = 0 \), the magnetic field can be written with straight field line coordinates in contravariant form:

\[ B = B_{\rho} \mathbf{e}_\rho + B_{\lambda} \mathbf{e}_\lambda = B(\rho, \lambda, Z) \quad \mathbf{B} = \frac{\nabla B}{\nabla B} \frac{\mathbf{B}^2}{\nabla B} \]

Substituting this form of the magnetic field into the equilibrium equations, the force balance can be written as [2]:

\[ \mathbf{r} \times \mathbf{B} = \nabla \times \mathbf{B} = \mu_0 J \quad \nabla \cdot \mathbf{B} = 0 \]

\[ F \equiv \mathbf{J} \times \mathbf{B} + \mathbf{F}_p = 0 \]

\[ F_p = 0 \]

Since \( F_p \) and \( \mathbf{F} \) are independent directions, \( F_p = 0 \) must hold in equilibrium. This yields the two constraint equations:

\[ F_p = 0 \quad \Rightarrow B = B(R, \phi, Z) \]

\[ F_p = 0 \quad \Rightarrow \phi = \phi(R, \phi, Z) \]

The basis vectors are functions of \( R, Z \) and their partial derivatives, so these two equilibrium conditions can be rewritten into equations for \( R(\rho, \theta, \lambda) \) and \( Z(\rho, \theta, \lambda) \). By discretizing \( R \) and \( \theta \) in the Poincaré sections, these become a system of ordinary differential equations (ODEs) with \( \lambda \) as the canonical “time”:

\[ x(C) = f(C, x) \]

\[ x(0) = x \]

\[ x(\lambda) = f(C, x) \]

\[ x(\lambda) = x \]

Comparison to VMEC
The computational domain is discretized using pseudo-spectral methods. For each Poincaré section, Fourier series in \( \theta \) and Chebyshev polynomials in \( \rho \) were used in this case [4]. The BVP is solved using a spectral collocation approach with a Fourier series interpolant in \( \lambda \).

Error Quantification
Below: Equilibrium force balance error \( F(\mathbf{B}/\mathbf{V}) \) for the BVP solution. The system of ODEs \( x = f(t, x) \) is equivalent to the equilibrium equations, so the approach directly minimizes the error at the collocation points. In other words, this pseudo-spectral method solves the equilibrium force balance at discrete points in space.

Future Work
- Refine the numerical methods to improve convergence and robustness to arbitrary inputs
- Investigate the existence of neighboring solutions through perturbation analysis
- Develop tools to assess the stability of equilibria
- Extend the formulation to include free-boundary equilibria
- Relax the constraint on nested flux surfaces to allow for magnetic islands and chaotic regions