Mechanical & Aerospace Engineering

3-D MHD Equilibria Construction based on a Poincaré Sections Approach

Daniel Dudt, Egemen Kolemen (ddudt@pppl.gov)

Department of Mechanical & Aerospace Engineering, Princeton University

PPPL

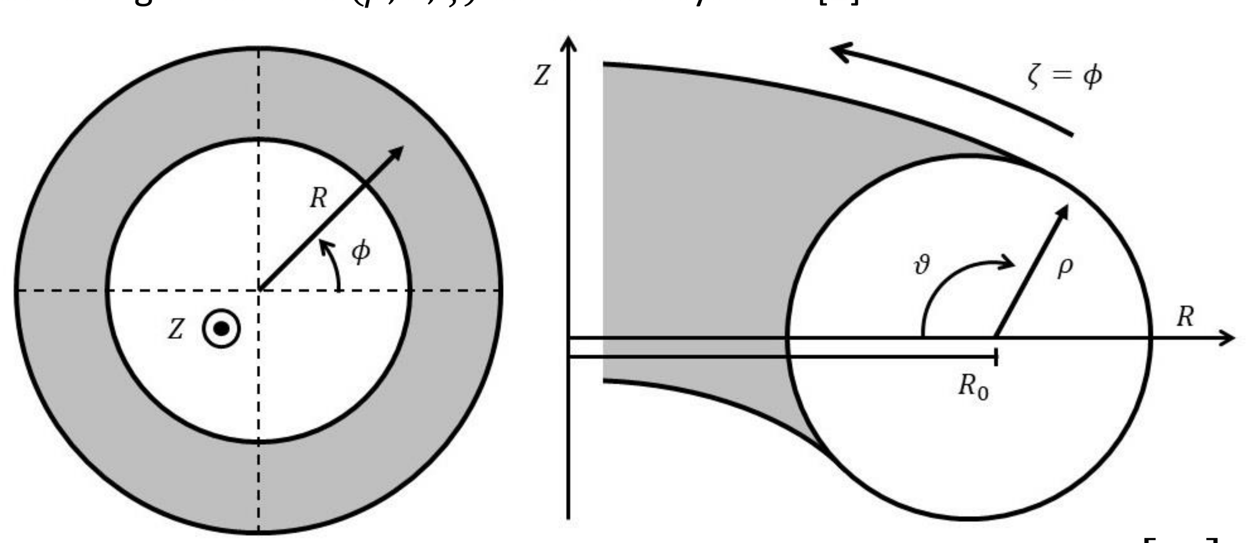
Motivation

Accurate and efficient equilibria calculations in fusion devices are crucial for understanding plasma science, improve diagnostics, avoiding disruptions, and enabling control systems. Conventional equilibria codes for non-axisymmetric geometries are computationally expensive and can return results that are poor solutions to the equilibrium force balance. We have developed a new code that directly solves the force balance equations and has the potential to converge faster than energy minimization approaches. As modern stellarator experiments continue to show a promising path to fusion, the development of advanced three-dimensional equilibria codes is becoming increasingly important.

Ideal MHD Equilibrium

Under the ideal MHD model of plasmas, the equilibrium condition reduces to a momentum balance, Ampere's Law, and the absence of magnetic monopoles [1]: $\boldsymbol{J}\times\boldsymbol{B}=\nabla p \qquad \nabla\times\boldsymbol{B}=\mu_0\boldsymbol{J} \qquad \nabla\cdot\boldsymbol{B}=0$

An equilibrium is typically constructed by prescribing the pressure profile $p(\rho)$ then numerically solving for the magnetic field $\boldsymbol{B}$ that satisfies the equilibrium conditions. The following diagram shows the toroidal (R,ϕ,Z) and straight field line (ρ,ϑ,ζ) coordinate systems [2]:



$$\sqrt{g}=\boldsymbol{e}_\rho\cdot\boldsymbol{e}_\vartheta\times\boldsymbol{e}_\zeta=R\left(R_\rho Z_\vartheta-R_\vartheta Z_\rho\right) \qquad \vartheta=\theta+\lambda \qquad \rho=\sqrt{\Psi_N} \qquad \boldsymbol{e}_\alpha=\begin{bmatrix}R_\alpha\\ \phi_\alpha\\ Z_\alpha\end{bmatrix}$$

Conventional Approach

The existence of flux surfaces and the rotational transform profile provide two additional constraints on the equilibrium problem:

$$\boldsymbol{B}\cdot\nabla p=0 \qquad \iota=\frac{B^\vartheta}{B^\zeta}$$

Most equilibria construction codes, including VMEC, solve for an equilibrium by minimizing the total energy of the plasma [3]:

$$W=\int\left(\frac{|B|^2}{2\mu_0}+p\right)dV$$

Poincaré Sections Approach

Instead of minimizing the total energy, the equilibrium force balance is used as a constraint to derive a system of ordinary differential equations that describe the "flow" of flux surfaces around the torus. The force balance can be written in the form [3]

$$(\nabla\times\boldsymbol{B})\times\boldsymbol{B}-\mu_0\nabla p=\boldsymbol{0}$$

$$\boldsymbol{F}\equiv F_\rho\nabla p+F_\beta\boldsymbol{\beta}=\boldsymbol{0}$$

$$F_\rho=\frac{1}{\mu_0}\left(B^\vartheta\left(\partial_\rho B_\vartheta-\partial_\vartheta B_\rho\right)+B^\zeta\left(\partial_\rho B_\zeta-\partial_\zeta B_\rho\right)\right)+p'$$

$$F_\beta=\frac{1}{\mu_0\sqrt{g}}\left(\partial_\vartheta B_\zeta-\partial_\zeta B_\vartheta\right) \qquad \boldsymbol{\beta}=\sqrt{g}\left(B^\zeta\nabla\vartheta-B^\vartheta\nabla\zeta\right)$$

Since ∇p and $\boldsymbol{\beta}$ are independent directions, the condition $F_\rho=F_\beta=0$ must hold in equilibrium. This yields two additional equations:

$$F_\rho=0\Rightarrow\partial_\zeta B_\rho=\iota\left(\partial_\rho B_\vartheta-\partial_\vartheta B_\rho\right)+\partial_\rho B_\zeta+\mu_0\frac{p'}{B^\zeta}$$

$$F_\beta=0\Rightarrow\partial_\zeta B_\vartheta=\partial_\vartheta B_\zeta$$

Satisfying the divergence-free field and assuming nested flux surfaces, the magnetic field can be written in the contravariant form

$$\boldsymbol{B}=B^\vartheta\boldsymbol{e}_\vartheta+B^\zeta\boldsymbol{e}_\zeta=B^\zeta\left(\iota\boldsymbol{e}_\vartheta+\boldsymbol{e}_\zeta\right) \qquad B^\zeta=\frac{\Psi'}{2\pi\sqrt{g}}$$

Substituting this form of the magnetic field into the two equilibrium conditions above yields equations for $\ddot{R}(\rho,\vartheta,\zeta)$ and $\ddot{Z}(\rho,\vartheta,\zeta)$. This formulation allows the task of solving for an equilibrium to be converted into a boundary-value problem (BVP) with a system of ordinary differential equations and periodic boundary condition:

$$\boldsymbol{x}(\zeta)=\begin{bmatrix}R(\rho,\vartheta)\\ Z(\rho,\vartheta)\\ \lambda(\rho=1)\\ \dot{R}(\rho,\vartheta)\\ \dot{Z}(\rho,\vartheta)\\ \dot{\lambda}(\rho=1)\end{bmatrix} \qquad \dot{\boldsymbol{x}}=\begin{bmatrix}\dot{R}\\ \dot{Z}\\ \dot{\lambda}\\ \frac{2\pi R}{\Psi'}\left(k_\rho Z_\vartheta-k_\beta Z_\rho\right)\\ -\frac{2\pi R}{\Psi'}\left(k_\rho R_\vartheta-k_\beta R_\rho\right)\\ \dfrac{\boldsymbol{e}_\vartheta\cdot\left(\boldsymbol{e}_{\theta\theta}\dot{\theta}^2+2\,\boldsymbol{e}_{\theta\phi}\dot{\theta}+\boldsymbol{e}_{\phi\phi}\right)-\frac{k_\beta}{B^\zeta}}{\boldsymbol{e}_\vartheta\cdot\boldsymbol{e}_\vartheta}\end{bmatrix}=f(\zeta,\boldsymbol{x})$$

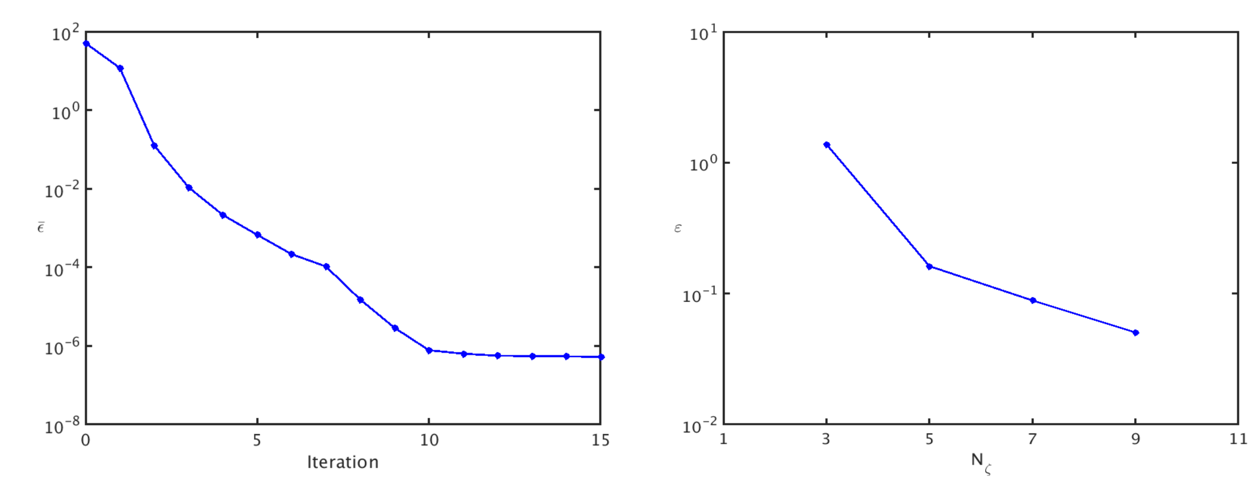
$$\boldsymbol{x}(0)=\boldsymbol{x}(2\pi)$$

$$k_\rho=\iota\left(\partial_\rho B_\vartheta-\partial_\vartheta B_\rho\right)+\partial_\rho B_\zeta+\mu_0\frac{p'}{B^\zeta}$$
$$-\partial_\zeta B^\zeta\left(\iota\boldsymbol{e}_\rho\cdot\boldsymbol{e}_\vartheta+\boldsymbol{e}_\rho\cdot\boldsymbol{e}_\zeta\right)-B^\zeta\left(\boldsymbol{e}_{\rho\zeta}\cdot\left(\iota\boldsymbol{e}_\vartheta+\boldsymbol{e}_\zeta\right)+\iota\boldsymbol{e}_\rho\cdot\boldsymbol{e}_{\vartheta\zeta}\right)$$
$$k_\beta=\partial_\vartheta B_\zeta-\partial_\zeta B^\zeta\left(\iota|\boldsymbol{e}_\vartheta|^2+\left(\boldsymbol{e}_\vartheta\cdot\boldsymbol{e}_\zeta\right)\right)-B^\zeta\boldsymbol{e}_{\vartheta\zeta}\cdot\left(2\iota\boldsymbol{e}_\vartheta+\boldsymbol{e}_\zeta\right)$$

The variable λ is used instead of R and Z at the last closed flux surface where the fixed-boundary $R(\theta)$, $Z(\theta)$ is given as a constraint. The unknows $\boldsymbol{x}$ are defined on a Poincaré section at a given toroidal angle ζ. The periodic boundary condition arises since the equilibrium Poincaré section must represent the same flux surfaces after integrating over a full field period.

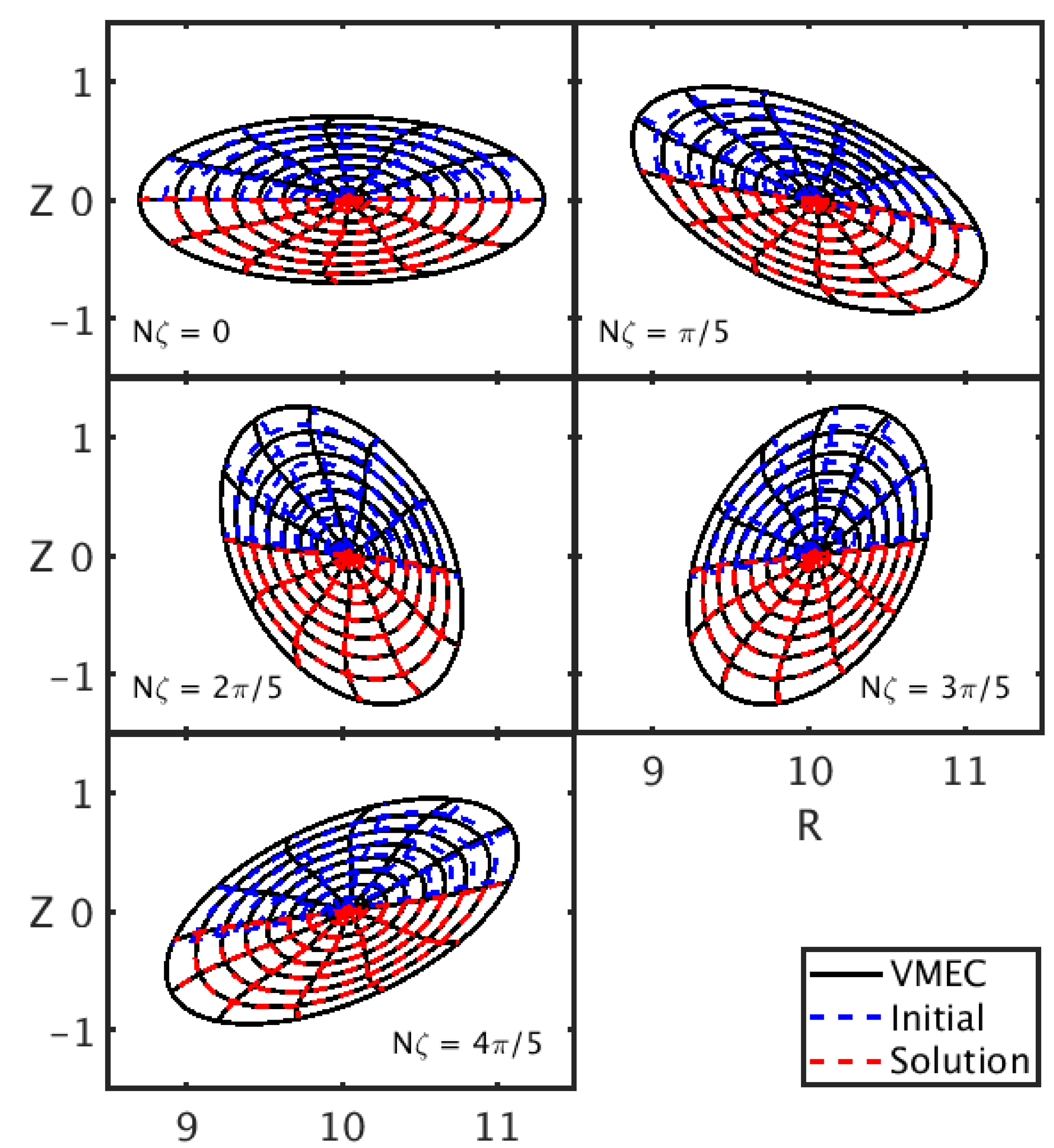
Comparison to VMEC

The computational polar domain is discretized using pseudo-spectral methods with a Fourier series in ϑ and Chebyshev polynomials in ρ [4]. The BVP is solved using a spectral collocation approach with a Fourier series expansion in ζ.



Left: Normalized sum of squares of the residuals at each iteration of the Levenberg–Marquardt algorithm used to solve the BVP.

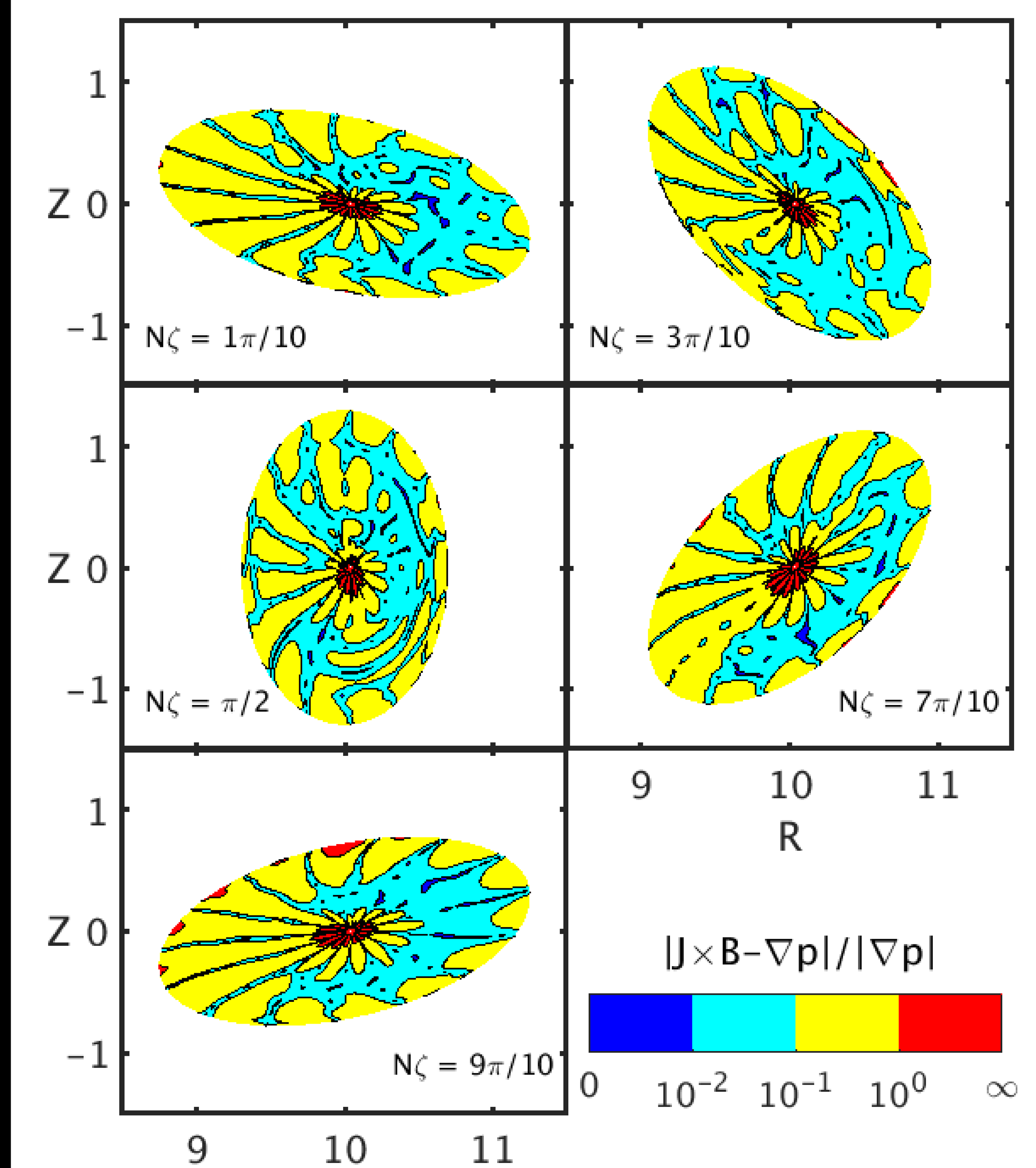
Right: $|\boldsymbol{x}(0)-\boldsymbol{x}^{VMEC}(0)|$ as a function of the number of Fourier modes used in the toroidal expansion (number of Poincaré sections).



Above: Comparison of flux surfaces between the BVP solution and VMEC equilibrium. The initial guess for the solution was obtained by scaling the boundary surface proportional to ρ.

Error Quantification

Below: Equilibrium force balance error $|\boldsymbol{F}|/|\nabla p|$ for the BVP solution. The system of equations $\dot{\boldsymbol{x}}=f(\zeta,\boldsymbol{x})$ is equivalent to the equilibrium constraints, so the approach directly minimizes the error at the collocation points. The midpoints of the toroidal angles where $\boldsymbol{x}(\zeta)$ is defined are shown.



All the results presented are for a heliotron with the following boundary and profile inputs:

$$R=10-\cos\theta-0.3\cos(\theta-N\phi) \qquad Z=\sin\theta-0.3\sin(\theta-N\phi)$$

$$p(\rho)=p_0\left(1-\rho^2\right) \qquad \iota(\rho)=1.618$$

with the number of field periods $N=19$ and p_0 corresponds to $\beta=1.1\%$. A basis of 8 Fourier modes and 8 Chebyshev polynomials were used.

Future Work

- Refine numerical methods to improve accuracy and convergence rate
- Develop tools to investigate stability of equilibria and the existence of neighboring solutions (see other poster)
- Restructure the boundary condition for free-boundary equilibria
- Formulate a method to allow for magnetic islands and chaotic regions

[1] Freidberg, J. (2014) [2] D'haeseleer, W. D., Hitchon, W. G., Callen, J. D., & Shohet, J. L. (1991) [3] Hirshman, S. P., & Whitson, J. C., (1983) [4] Boyd, J. (2000)