Comparison of the DESC and VMEC 3D Equilibrium Codes

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Stellarator Optimization - VMEC

$$\frac{\partial X_j^{mn}}{\partial t} = F_j^{mn}$$

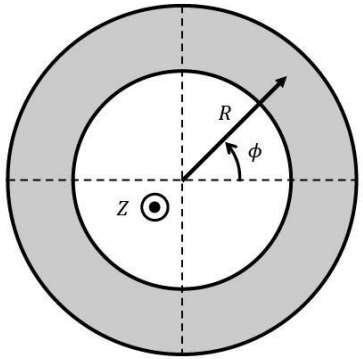
$$W = \int_V \left(\frac{B^2}{2\mu_0} + \frac{p}{\gamma - 1} dV \right)$$

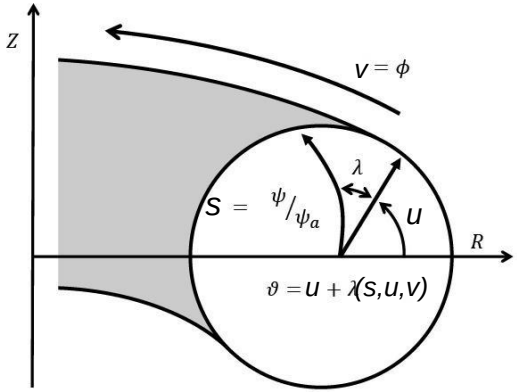
- Spectral inverse equilibrium code (Hirshman and Whitman, 1983)
  - Solves 'inverse' equilibrium problem
  - want location of flux surfaces R,Z as function of computational domain (**s**,**u**,**v**)
- Fourier series on flux surfaces, defined only on discrete flux surfaces
- Angular derivatives analytic, but radial derivatives are finite difference
- Minimizes energy with steepest-descent method based on variational principle

$$R(s,u,v) = \sum_{m=0,n=-N}^{M,N} R_{mn,c}(s)cos(mu - nvN_{FP}) + R_{mn,s}(s)sin(mu - nvN_{FP})$$

$$\lambda(s,u,v) = \sum_{m=0,n=-N}^{M,N} \lambda_{mn,c}(s)cos(mu - nvN_{FP}) + \lambda_{mn,s}(s)sin(mu - nvN_{FP})$$

$$Z(s,u,v) = \sum_{m=0,n=-N}^{M,N} Z_{mn,c}(s)cos(mu - nvN_{FP}) + Z_{mn,s}(s)sin(mu - nvN_{FP})$$





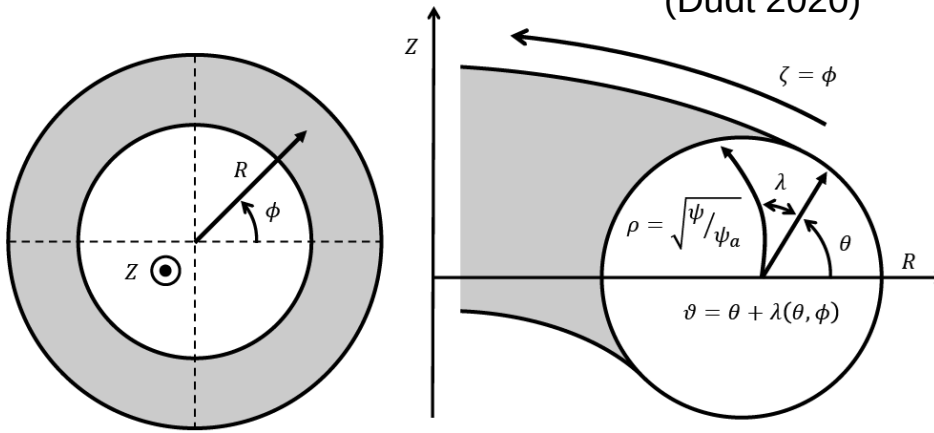
Stellarator Optimization - DESC

- Pseudospectral 3D Equilibrium Code
  - 3-D Fourier-Zernike spectral representation
- Solve for equilibrium by minimizing force error directly at collocation nodes
- Uses Newton-Raphson type methods to solve the resulting system of nonlinear equations

$$R(\rho,\theta,\zeta) = \sum_{m=-M,n=-N,l=0}^{M,N,L} R_{lmn}\mathcal{Z}_l^m(\rho,\theta)\mathcal{F}^n(\zeta)$$
$$\lambda(\rho,\theta,\zeta) = \sum_{m=-M,n=-N,l=0}^{M,N,L} \lambda_{lmn}\mathcal{Z}_l^m(\rho,\theta)\mathcal{F}^n(\zeta)$$
$$Z(\rho,\theta,\zeta) = \sum_{m=-M,n=-N,l=0}^{M,N,L} Z_{lmn}\mathcal{Z}_l^m(\rho,\theta)\mathcal{F}^n(\zeta)$$

$$\boldsymbol{F} = \boldsymbol{J} \times \boldsymbol{B} - \nabla p = 0$$

(Dudt 2020)



Force Error is Calculated from VMEC Starting with R,Z,λ

- Read in Fourier coefficients from VMEC wout file
  - convert λ from half -> full mesh
- Find necessary angular derivatives analytically
- Find necessary radial derivatives numerically
  - finite difference, splines, etc.
- Multiply out in real space to find force error **F**
- VMEC Error metric taken as volume-averaged force error over (s=0.1 - s=0.99) to avoid edge/axis
  - Normalized by volume average **∇**p

$$R(s,u,v), Z(s,u,v), \lambda(s,u,v)$$

$$\mathbf{e}_s = \begin{bmatrix} \partial_s R \\ 0 \\ \partial_s Z \end{bmatrix} \quad \mathbf{e}_u = \begin{bmatrix} \partial_u R \\ 0 \\ \partial_u Z \end{bmatrix} \quad \mathbf{e}_v = \begin{bmatrix} \partial_v R \\ R \\ \partial_v Z \end{bmatrix}$$

$$\sqrt{g} = \mathbf{e}_s \cdot \mathbf{e}_u \times \mathbf{e}_v$$

$$B^u = \frac{1}{\sqrt{g}}\left(\chi' - \psi' \frac{\partial \lambda}{\partial v}\right) \qquad B^v = \frac{1}{\sqrt{g}}\psi'\left(1 + \frac{\partial \lambda}{\partial u}\right)$$

$$J^s = \frac{1}{\mu_0 \sqrt{g}}\left(\frac{\partial B_v}{\partial u} - \frac{\partial B_u}{\partial v}\right)$$

$$J^u = \frac{1}{\mu_0 \sqrt{g}}\left(\frac{\partial B_s}{\partial v} - \frac{\partial B_v}{\partial s}\right)$$

$$J^v = \frac{1}{\mu_0 \sqrt{g}}\left(\frac{\partial B_u}{\partial s} - \frac{\partial B_s}{\partial u}\right)$$

$$\boldsymbol{B}(s,u,v), \boldsymbol{J}(s,u,v)$$

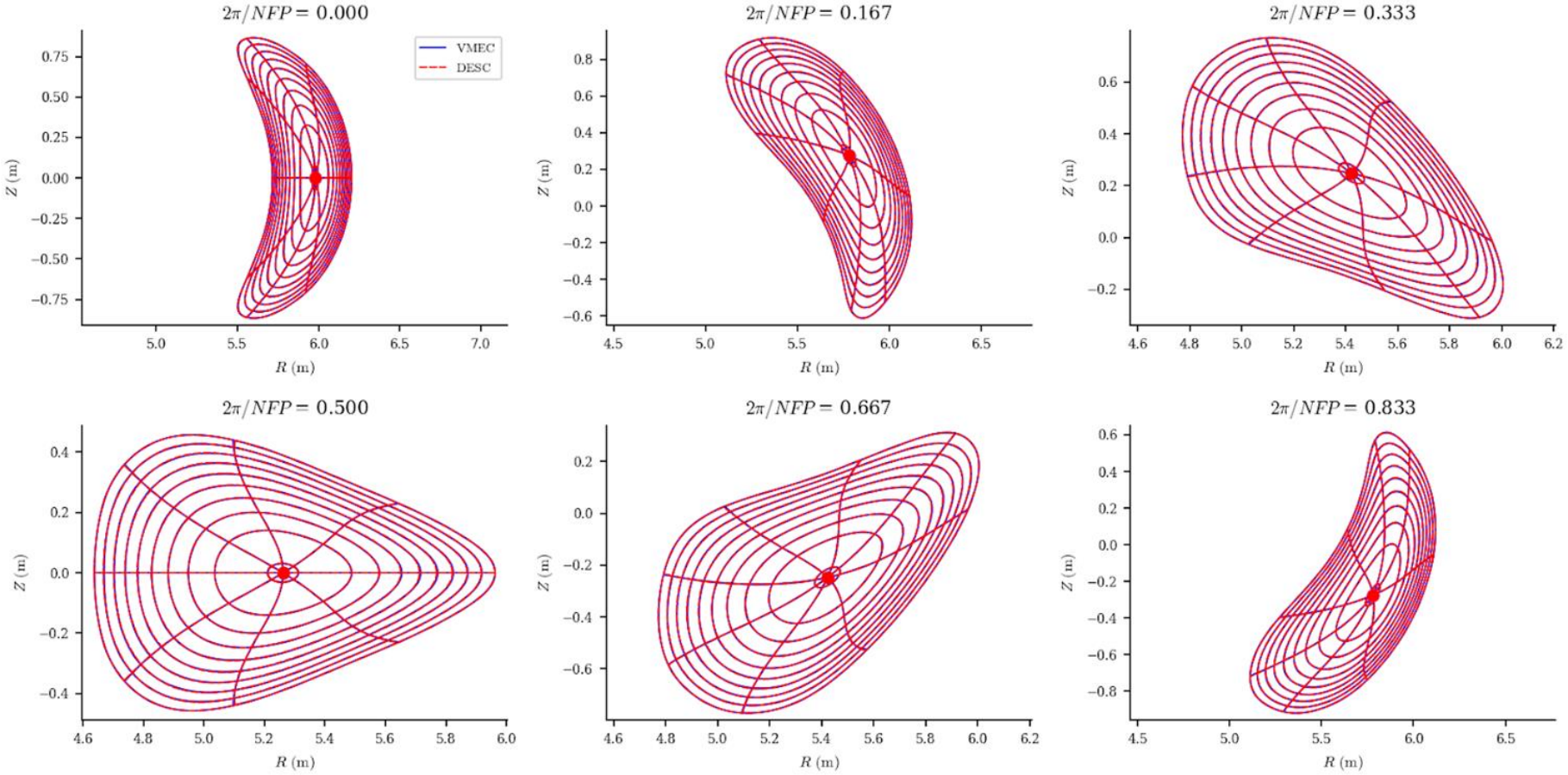
$$F_s = \sqrt{g}(J^v B^u - J^u B^v) + p'$$

$$F_\beta = J^s$$

$$\boldsymbol{F}(s,u,v)$$

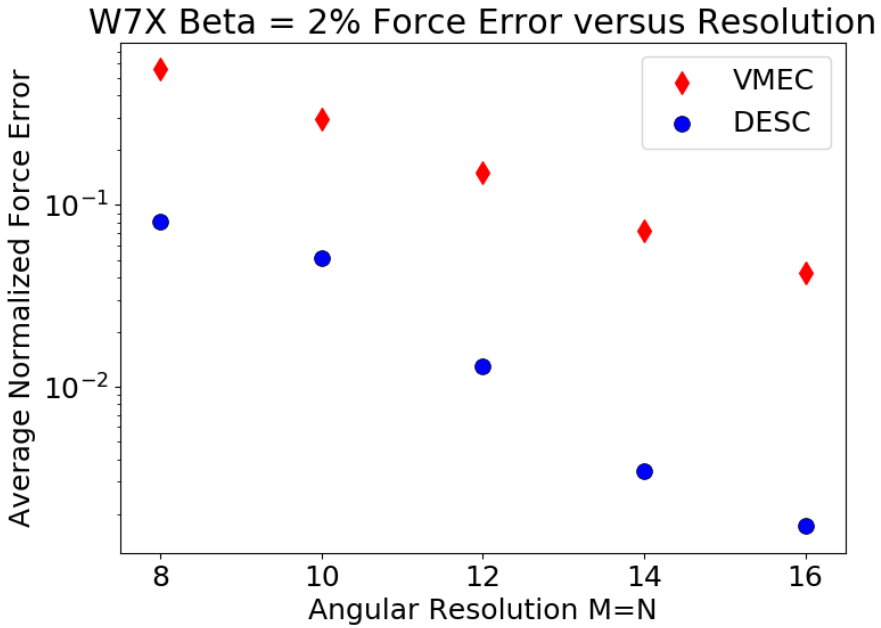
$$< F >_{vol} = \frac{\int_{\theta=0}^{2\pi}\int_{\phi=0}^{2\pi}\int_{s=0.1}^{0.99} |F||\sqrt{g}| ds d\phi d\theta}{\int_{\theta=0}^{2\pi}\int_{\phi=0}^{2\pi}\int_{s=0.1}^{0.99} |\sqrt{g}| ds d\phi d\theta} \qquad < |\nabla p| >_{vol} = \frac{\int_{\theta=0}^{2\pi}\int_{\phi=0}^{2\pi}\int_{s=0}^{1} |\nabla p||\sqrt{g}| ds d\phi d\theta}{\int_{\theta=0}^{2\pi}\int_{\phi=0}^{2\pi}\int_{s=0}^{1} |\sqrt{g}| ds d\phi d\theta}$$

Solution Comparison - Flux Surfaces Indistinguishable by Eye



For Given Resolution, DESC has lower Force Error

W7X M=N=12 (288 Fourier Modes)			
Code	Energy (J)	$\|\mathbf{F}\|/\|\nabla p\|$	Runtime (1 CPU)
DESC	4.23243e+08	0.0132	0.71h
VMEC	4.23243e+08	0.151	1.19h



For Given Time to Solution, DESC has lower Force Error

W7X M=N=12 (288 Fourier Modes)			
Code	Energy (J)	$\|\mathbf{F}\|/\|\nabla p\|$	Runtime (1 CPU)
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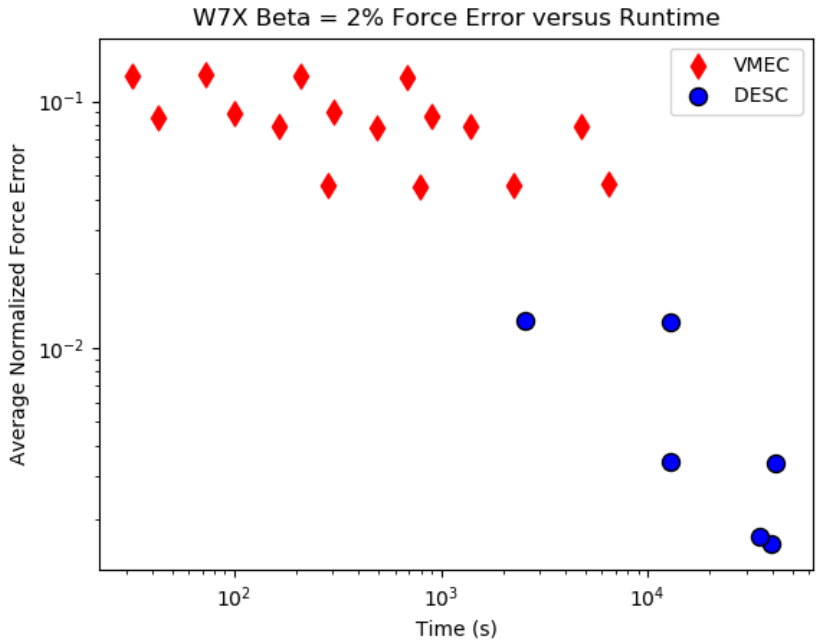
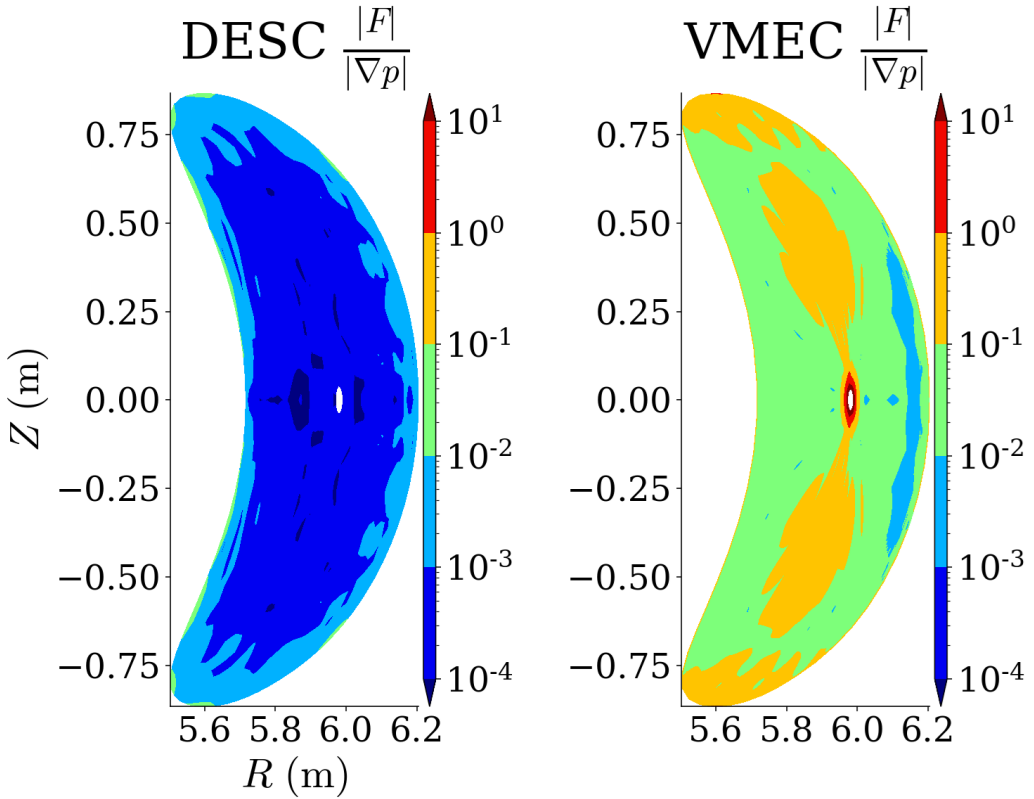


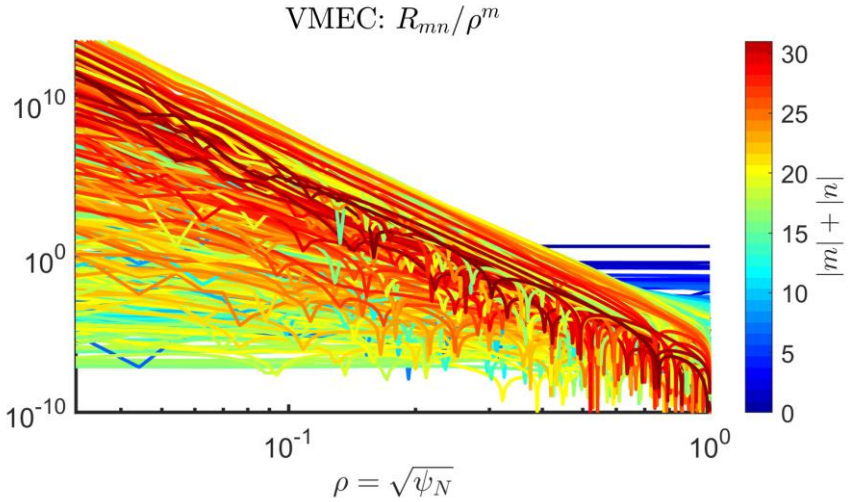
VMEC Force Error is Noticeably Higher Near-axis

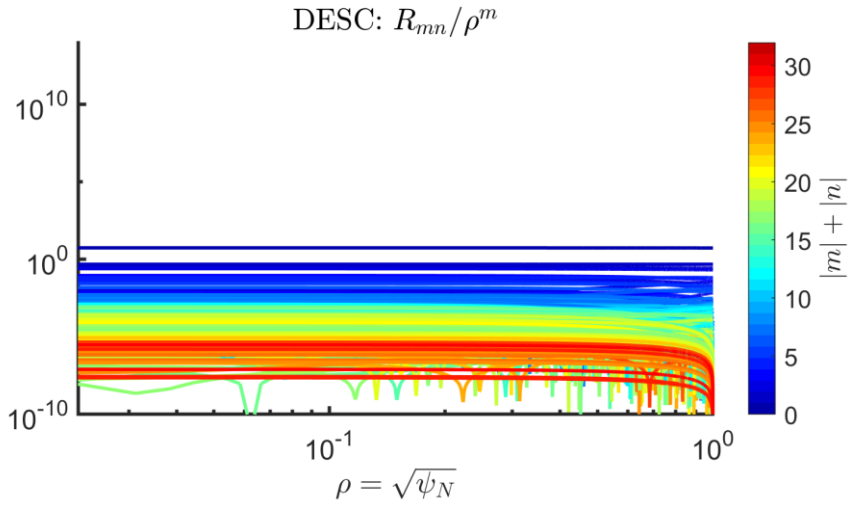
- This could be due to VMEC's Fourier coefficients not explicitly obeying analyticity constraint near axis



Analytic Constraint Near Axis

- Fourier coefficients of an analytic function must scale as ρ^m near axis (Lewis and Bellan 1990)
- DESC coefficients obey this inherently due to Zernike basis, VMEC do not, especially for higher order modes





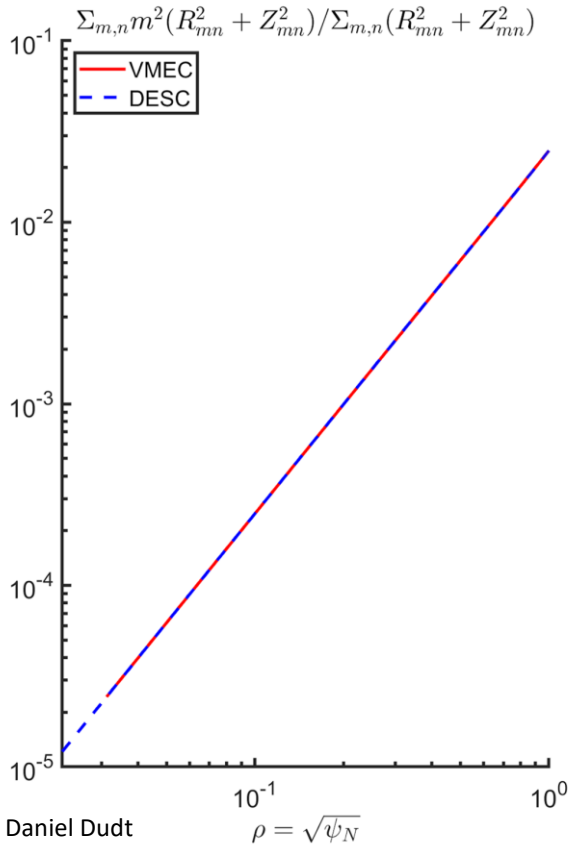
Plots Courtesy of Daniel Dudt

Both DESC and VMEC Poloidal Angle are Optimal

- Spectral condensation as defined by Hirshman and Meier (1985)

$$M(p,q) \equiv \frac{\Sigma_{m=1}\, m^q S_p(m)}{\Sigma_{m=1} S_p(m)}$$

- Minimization of M wrt poloidal angle corresponds to an optimally condensed Fourier spectrum -> explicit constraint in VMEC
- DESC poloidal angle found through optimization is as optimal as VMEC's



Plot Courtesy of Daniel Dudt

Conclusions and Future Work

- For given resolution, DESC is more accurate than VMEC in terms of force-error metric
  - energy of equilibrium state is essentially same as VMEC
- DESC's Zernike basis better treats the analyticity constraints at axis
  - Zernike radial basis has been adopted by SPEC code as well for innermost computational region (Qu et al 2020) for this quality
- DESC and VMEC poloidal angle choice are both optimal
- Comparison with DESC on GPU vs. PARVMEC is still underway

Thanks! Questions?

Check out these other DESC Presentations!

- **PP11.00087 : Improvements to the DESC code for finding and optimizing stellarator equilibria**
- **BO08.00010 : Stellarator Optimization with DESC**

Check out our Code!

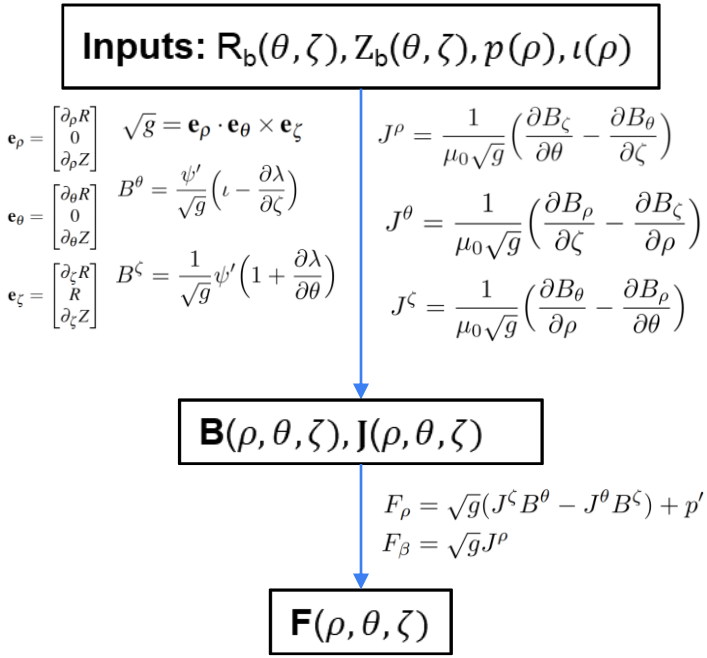
Publication: Dudt et. al. *Phys. Plasmas*. 2020.

Repository: https://github.com/PlasmaControl/DESC

Python Package: `pip install desc-opt`

Backup

What DESC Solves



R, Z, λ and their derivatives are evaluated on a collocation grid in (ρ, θ, ζ), then multiplied to calculate **F** on this grid

This leads to a system of equations comprised of the force balance error evaluated at the collocation nodes, which we want to make equal to zero -> Can use root-finding or least-squares to solve

$$\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}$$

x is the spectral coefficients of *R,Z,λ*, which is what we are changing to minimize **f**

DESC Fourier-Zernike Spectral Basis

$$R(\rho,\theta,\zeta) = \sum_{m=-M,n=-N,l=0}^{M,N,L} R_{lmn}\mathcal{Z}_l^m(\rho,\theta)\mathcal{F}^n(\zeta)$$

$$\lambda(\rho,\theta,\zeta) = \sum_{m=-M,n=-N,l=0}^{M,N,L} \lambda_{lmn}\mathcal{Z}_l^m(\rho,\theta)\mathcal{F}^n(\zeta)$$

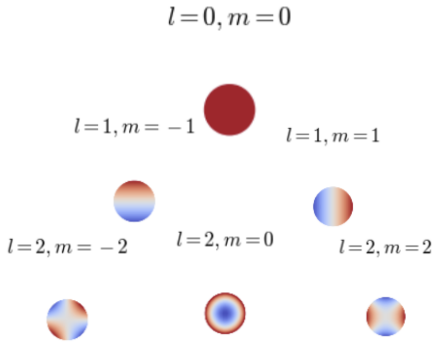
$$Z(\rho,\theta,\zeta) = \sum_{m=-M,n=-N,l=0}^{M,N,L} Z_{lmn}\mathcal{Z}_l^m(\rho,\theta)\mathcal{F}^n(\zeta)$$

$$\mathcal{Z}_m^l(\rho,\theta) = \begin{cases} \mathcal{R}_l^{|m|}(\rho)cos(|m|\theta) & \text{for } m \geq 0 \\ \mathcal{R}_l^{|m|}(\rho)sin(|m|\theta) & \text{for } m < 0 \end{cases}$$

$$\mathcal{R}_l^{|m|}(\rho) = \sum_{s=0}^{(l-|m|)/2} \frac{(-1)^s(l-s)!}{s!\,[(l+|m|)/2-s]!\,[(l-|m|)/2+s]!}\rho^{l-2s}$$

$$\mathcal{F}^n(\zeta) = \begin{cases} cos(|n|N_{FP}\zeta) \text{for } n \geq 0 \\ sin(|n|N_{FP}\zeta) \text{for } n < 0 \end{cases}$$

Zernike Basis in (r, θ)



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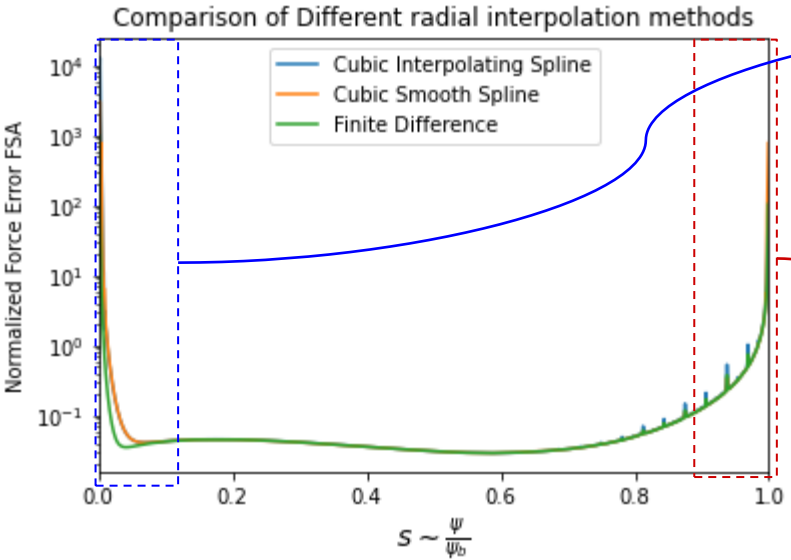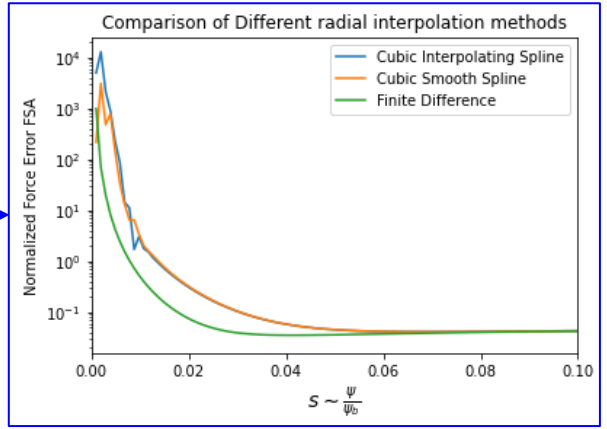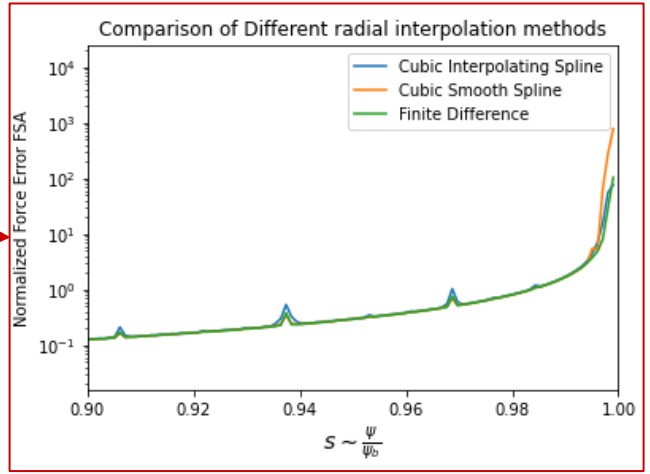
Radial Derivative Method Does Not Affect VMEC Force Error Significantly